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LINEAR PROGRAMMING AND ECONOMIC THEORY

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## Linear Programming and Economic Theory

### I Introduction

Anyone who is familiar with both economic theory and linear programming must admit that linear programming has been one of the most exciting developments in economic theory of recent years. A glance at the economic journals shows that a fairly extensive literature has already piled up even though the subject is scarcely old enough chronologically to go to kindergarten. Outside the realm of economics, there appear numerous linear programming articles in magazines of applied business practice and in more purely mathematical journals. But even these publications do not tell the whole story

, since a veritable gusher of unpublished research papers on the theory and application of linear programming seem to pour out in each current month.

This conference itself testifies to the widespread interest in the subject. And periodic conferences like this one, and like the original June, 1949 conference in Chicago, provide convenient bench-marks to measure our progress. The Washington Conference on Inequalities and Programming of June, 1951 showed that the theory of linear programming had been extended in the previous two years; and it, of course, revealed the tremendous quantitative expansion of applications. Similarly, just reading over the advance titles of the papers at this conference acquaints one with the further advances of the last few years.

Ordinally, therefore, we can truthfully say: Every day, in every way, we are getting better and better. <sup>¶</sup> For the purpose of decision-making, ordinal

comparisons are usually alone relevant. But if I were to digress momentarily to make an odious quasi-cardinal comparison, frankness would compel me to state: The Proceedings of the 1949 conference, Activity Analysis of Production and Allocation--which Tjalling Koopmans so beautifully edited--is the volume on my bookshelf that I find myself most often referring subsequent to; the primary advance recorded in the 1951 conference seemed to have been chiefly in the (no doubt important) field of computation and in the extension of applications. It is too soon to tell whether this same phenomenon of "diminishing returns" will be discernable in a numerical appraisal of the advance this conference will record.

Let me hasten to qualify this remark. Even at the risk of seeming to contradict the famous J. M. Clark aphorism that "everything but intelligence is subject to the law of diminishing returns," we must recognize that most conscious direction of scientific research must be toward pushing it along paths of diminishing returns. So there should be nothing discouraging in my comparison. More than that: my evaluation should stand as a tribute to the fundamental work of the late 1940's rather than as a denigration of subsequent developments. Finally, I have given an economic theorist's subjective evaluation. I can readily imagine that according to the indifference curves of some applied mathematicians, the subject is just now beginning for the first time to become really interesting.

## II Sources of Inspiration for Linear Programming

The richness of linear programming for the economic theorist can be illustrated by the dilemma that we have been facing in preparing an expository RAND monograph aimed to introduce the non-specialist economist to the subject. Some years ago I prepared some introductory chapters, written from

the general viewpoint of the economist. But with each passing month the projected outline grew and grew in length. This was not because of any desire to make the treatment comprehensive, but rather simply a reflection of the many interesting topics where economic theory and linear programming overlap. I began to feel like Tristram Shandy, who, you will recall, took several years to write up the first three weeks of his life. To make the series converge to a finite sum, Professors Robert Dorfman and Robert Solow have joined the project, but they too have had to face its tendency to grow out of bounds.

What are some of the important areas where economic theory and linear programming overlap? I don't think we can do better than list the four sources of inspiration for linear programming given by Professor Koopmans in his Introduction to the Activity Analysis book. These are as follows:

(1) The recognition in the early 1930's by such continental economists as Meissner, Von Stackelberg, Zeuthen, Schlesinger, v. Neumann, and Wald that the simple Cassel version of Walrasian general equilibrium could not be adequately appraised by an uncritical counting of the number of its equations and the number of its unknowns.

(2) The "new welfare economics" in the various versions of Lerner, Bergson, Kaldor, Hicks, Samuelson, Lange, and Arrow which <sup>threw</sup> new light on the earlier writings of Smith, Walras, von Wieser, Marshall, Pareto, Barone, Pigo, von Mises, and Fred Taylor.

(3) The interindustry input-output theories and measurements of Leontief, Evans, and Hoffman; and the related multi-sector analysis of Keynes, Harrold, Salant, Machlup, Metzler, Solow, Goodwin, and Chipman.

(4) The specific programming and optimizing problems raised by defense and military problems--and, more generally, the numerous optimizing problems

- 4 -

that business firms have always had to solve in their quest for profits and survival. (The Cornfield-Stigler diet problem and the Hitchcock-Koopmans-Kantorovich transportation problem might be put in this category.)

I think to these four sources of inspirations should be added at least five more, even though only the last two of these are primarily economic in nature.

(5) The interrelations between linear programming and the theory of games of Borel, v. Neumann, R. A. Fisher, and Morgenstern.

(6) The foundations of the Wald statistical decision theory, casting now light on the Fisher and Neyman-Pearson theories of statistics; and the related foundations of personal probability and decision making of the Ramsey-Savage type.

(7) The purely mathematical interest of Minkowski, Weyl, Bonnesen and Fenchel, and a host of other mathematicians in various aspects of the theory of convex sets and of topology generally.

(8) Within the area of economic maximizing problems, theorists during the last couple of decades had begun to concentrate on the inequalities that characterize a maximum rather than on the first-derivative equivalences that happen to characterize certain smooth interior maximum points. The economic theories of index numbers, of revealed preference, of Le Chatelier principles, all these are examples of this trend antedating the birth of linear programming.

(9) Finally, in the economic theories of arbitrage, speculation, location, and rationing there have from the earliest days been recognized to enter problems of inequalities; since these problems have not always been formulated as maximum or as general equilibrium problems, I have added them as a category

- 5 -

separate from the earlier categories. (The names of Ricardo, Cournot, Scitovsky, de Graaf and other economists could be specified in this regard.)

### III Theoretical Insights Provided by Linear Programming

If you examine the listed ways that economic theory has inspired linear programming, you may superficially infer that the process has been one of unilateral causation from economics to programming. Inasmuch as economic theory antedates linear programming as a formal discipline of study and research, this is in a sense natural and to be expected. However, you may not infer that programming is the inferior activity even where it has taken its problems directly from ancient economic theory. Often, the economist has had the important, but vastly easier task, of asking certain questions; and the linear programming expert has often been able to answer those questions--in some cases to give answers to them for the first time.

I have been wracking my brain to see if I could think of any converse example--where linear programming has raised new and important questions for the economic theorist that he had not previously thought about. I have not offhand been able to produce any and I would hope that some economist or mathematician will provide us with some such examples for discussion.<sup>1</sup>

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<sup>1</sup>In the oral discussion to this paper, Dr. Martin Beckmann suggested that linear programming had raised a number of interesting questions for the theorist of locational problems. And as I shall argue later, many of the general mathematical tools that are needed for linear programming--such as convex sets, fixed point theorems, saddlepoints, etc.--turn out to be extremely useful to the modern mathematical economist.

Upon reflection, I must admit that linear programming has succeeded in tackling empirically many of the general problems that the theorist had always talked about. Thus, the economist speaks glibly of a multi-product firm as having a cost function dependent upon all its outputs and as reaching equilibrium when it equates each product's marginal revenue to its marginal cost. Work of



- 6 -

Cooper, Charnes, Manne, Henderson, Schlaifer, and others have given concrete applications to what might otherwise remain in the category of "empty boxes."

When Dr. George Dantsig asked me in the oral discussion why economic theorists are so "uninterested" in linear programming applications, I replied: "Theorists are congenitally a little bored with concrete applications. They prefer to consider the general qualitative aspects of things rather than themselves to become interested in the quantitative details of, say, the oil industry's multiple products. Moreover, the theorist often suspects that the linear programmer grinds out an exact solution to a rather idealized approximation to the true reality--so that in any case only the qualitative direction of changes can be inferred from the programming results." Upon reflection, I feel that parts of my answer are indefensible. The theorist should be interested in concrete applications--if they are valid. Also, if businessmen come increasingly to use linear programming techniques--even when not valid--then the theorist must take this fact into account in describing their behavior, in the same way that he takes into account the systematic aberrations of widely used accounting techniques. This does not mean that every economic theorist must himself specialize in solving problems of internal administration for firms or other maximizing units: such applications, when they become coherent enough, will tend to move outside the narrow discipline of economics in the same way that accounting and technology have done.

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Does this failure to supply questions for the economic theorist imply that linear programming is, from his selfish point of view, sterile? Not at all. Programming theory has not only provided the theorist with many answers to his questions. It has also provided him with fairly rigorous proofs for some of his theorems--or as the jurist would say--for some of his conjectures. Even more important it has provided him with feed-back insights into the fundamentals of his subject.

Thus, the modern economist had, prior to the birth of linear programming as a recognizable separate entity, attained a pretty fair understanding of the nature of a pricing mechanism for the attainment of various welfare-economics optima. In other words he has made considerable progress beyond Adam Smith's notion of the invisible hand, toward a deeper understanding of what that notion involved. None the less, no one who understands both economic theory and programming theory is likely to deny that the latter's fundamental duality theorems

- 7 -

have added to his understanding of the pricing mechanism and its limitations.

#### IV Existence of Competitive Equilibrium

On this same afternoon's program, Professors Harold Kuhn, a mathematician, and Lionel McKenzie, an economist, are discussing the problem which I listed as one of the first sources of inspiration for linear programming. Until Wald's proof came along, economists had no rigorous demonstration of the existence of competitive equilibrium. Indeed some incautious formulations of the Walrasian system, such as that of Cassel, gave rise to long unnoticed contradictions and difficulties. The keen literary economist—and he does exist—always realized that the way out of these contradictions came from making some factors free and then dropping the requirement that all of a free factor be employed, which is precisely the mathematician's final resolving of the paradox. (On this same afternoon's program, Professor Georgesou-Roegen, both a keen literary and mathematical economist, interprets the economic history of Roumania in terms of such a redundancy of labor; the Dutch economist Valk offered a similar hypothesis to explain depression unemployment, and my colleagues at the MIT Center for International Studies, Dr. Ronenstern-Rodan and Dr. Pokaus have suggested similar interpretations to explain the redundancy of labor in modern Italy.)

The nature of the difficulty with the Cassel system is easy to see from the following two sets of equations

$$\begin{aligned}
 (1) \quad & \sum_{j=1}^n a_{ij} X_j = V_i & (i=1,2,\dots,n) \\
 (2) \quad & \sum_{i=1}^n a_{ji} W_i = P_j & (j=1,2,\dots,n)
 \end{aligned}$$

- 8 -

where  $X$ 's are  $n$  outputs with  $P$ 's their competitive market prices, and  $V$ 's are  $m$  factor inputs and  $W$ 's their market factor prices, and where  $a_{ij}$  are specified non-negative fixed coefficients of production. Equations (1) say that all factors are used up and (2) say that all goods sell at their unit costs of production, with competition grinding out all profits or surpluses.

Even with  $n \neq m$ , there is nothing contradictory about (1) and (2) until the theorist goes on to make the assumption that the factor supplies, as given by the right-hand  $V$ 's in (1) can be arbitrarily specified, at the same time that the commodity prices on the right of (2) are all arbitrarily specified. Usually, we think of the number of goods as exceeding the number of factors, so  $n > m$ . This means that the set (1) is under determined, with  $n-m$   $X$ 's being capable of taking on arbitrary values.

This is troublesome, but not logically contradictory. However, look at (2). With all  $P$ 's arbitrarily specifiable, we have  $n$  conditions on  $m < n$  unknown  $W$ 's. So (2) is overdetermined, possessing no solution. The degree of its overdeterminacy is, so to speak,  $n-m$ .

How was this basic irreconcilability overlooked? In part, because theorists counted the total number of unknowns in (1) and (2), which worked out to be  $n+m$  and found them equal to the total number of equations in (1) and (2), also  $n+m$ . So to speak, they unknowingly cancelled out the underdeterminacy of (1) taken by itself against the over determinacy of (2) taken by itself. This is logically illegitimate. This is also shown by the mathematical fact that the determinant

- 9 -

$$\left| \begin{array}{cccc} a_{11}a_{12} & \dots & a_{1n} & 0 & \dots & 0 \\ a_{m1}a_{m2} & \dots & a_{mn} & 0 & \dots & 0 \\ 0 & \dots & 0 & a_{11} & \dots & a_{m1} \\ \vdots & & & & & \\ 0 & \dots & 0 & a_{1n} & \dots & a_{mn} \end{array} \right| = 0 \text{ if } m \neq n$$

as Laplace's development shows.

Economists such as Wieser and Fred Taylor had avoided this difficulty by assuming that  $m = n$ . However, this does not avoid the difficulty; it only postpones the logical contradiction. For, in the first place, even with  $n = m$ , there is no reason at all why the  $a_{ij}$  matrix should not be singular: why shouldn't two goods use exactly the same proportion of inputs? If the theorist objects that this is equivalent to defining them as the same goods and therefore reducing  $n$  to one below  $m$ , this simply confesses that there is a logical difficulty.

Let us for the purpose of the argument suppose that Nature is kind and does give us a non-singular square matrix of  $a$ 's. Except in the trivial case where each good requires but one factor which is unique to it, (1) and (2)'s determinate solution for each prescribed set of positive  $V$ 's and  $P$ 's cannot have the property of always avoiding negative values. I.e. we can easily specify arbitrary positive values of the  $V$ 's and  $P$ 's which cause one or more of our economic unknowns to be negative. (This follows from simple economic reasoning if we supply the factors in proportions more extreme than any good uses; or it can be proved by the mathematical fact that a non-singular matrix of non-negative coefficients that is not the identity matrix must have negative elements in its inverse, so that  $a^{-1}V$  can for appropriate choice of positive

- 10 -

V's be made to have one or more negative element.)

Economists eventually learned all this, and if they had been numerically minded, they might have learned this even earlier. But one way that they had of resolving any such difficulties was to deny one of the postulates giving rise to the trouble: why keep insisting that the factor supplies  $V_i$  were prescribable at arbitrary levels? Why keep insisting that any competitive prices could be prescribed? Alternatively, why not prescribe that only those prices compatible with cost of production are possible? And only those factor supplies that can all be used?

Thus, we may still be able to avoid logical inconsistency by insisting on the equalities of (1) and (2) but letting all variables be unknowns--not just half the variables. Thus, we have  $n + m$  equations binding  $2(n-m)$  unknowns and there need be no overdeterminacy. There is of course underdeterminacy, but we feel that we can add taste or demand equations and disutility or factor supply equations that will serve to determine our system.

But do we know this? The economist feels intuitively that this is so; yet the mathematician will require proof of the Wald or other type. I shall briefly sketch the elements of such a proof, making slightly stronger assumptions than Wald does. But since I believe his assumptions are overstrong, from the economic viewpoint, there will not really be much difference between my assumptions and those of Wald and Schlesinger.<sup>2</sup>

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<sup>2</sup>Wald assumes that market demand functions relating totals bought by everybody satisfy what we today call the "Weak Axiom" of preference theory. Such an axiom holds for a single individual, but it is arbitrary to assume it holds for the market totals. Many plausible examples can be given of this fact. In a yet unpublished paper, I have proved that something like this--and more--would hold in a "good society" where incomes are always optimally redistributed

- 11 -

so as to maximise a social welfare function. Such a good society acts like a single individual--so we might as well from the beginning talk of Robinson Crusoe. Note that McKensie's proof of existence is free from this limitation; but he rightly abandons Wald's attempt to prove what is untrue of competitive equilibrium generally--namely uniqueness of equilibrium.

First, assume that there is a single Robinson Crusoe with regularly convex indifference curves describable by a smooth ordinal utility indicator  $U(X_1, \dots, X_n) = U(X)$  with the property: if  $U(A) = U(B)$ , then  $U(A+B) \geq U(A) = U(B)$ . The substitution ratios,  $\partial U / \partial X_1 / \partial U / \partial X_j$  are determinate functions  $f_{1j}(X)$  of the goods consumed, independently of the utility indicator. Finally, assume that Crusoe is indifferent as to how much of each  $V_j$  he supplies between 0 and  $\bar{V}_j$  where the latter are prescribed positive numbers.

These conditions will be sufficient to define a competitive equilibrium, which will

Maximize  $U(X_1, \dots, X_n)$  subject to

(3)

$$\sum_1^n a_{1j} X_j \leq \bar{V}_1 \quad (i=1, \dots, m)$$

Writing the resulting maximized value of  $U$  as  $F(\bar{V}_1, \dots, \bar{V}_m)$ , we can determine the resulting factor prices  $\bar{w}_j$  as proportional to  $\partial F / \partial \bar{V}_j$ , and prices  $P_i$  will be proportional to  $\partial U / \partial X_i$ , and will satisfy a set of relations just like (2) but with inequalities inserted to take account of the possibility that for goods not produced, price may exceed unit cost of production. Wald does admit the existence of inequalities in (1), but following Schlesinger he makes the unnecessarily restrictive assumption that if any  $X_i$  is zero, the resulting level of well-being is less than it is for any point at which all



- 12 -

$X$ 's are positive: this insures that every  $X_i$  is positive and that all the equalities hold in (2). Since uniqueness is arbitrary, I do not follow Wald in assuming that the rank of  $a$  is  $m$ ; it can be anything.

Time does not permit me to dwell on the naturalness of the fixed-point types of proof of the existence of competitive equilibrium. To some these may seem like rather sophisticated mathematical tools for the economist to be using; but to my mind, they do strikingly employ the economist's intuitive feeling that equilibrium intersections must exist if all the supply and demand functions have the appropriate continuity properties.

I heartily approve of the gentlemen's agreements that are made about continuity so that these beautiful theorems and proofs can be brought in. None the less, from the strict economics of the case, we must be prepared to encounter phenomena that lead away from the existence of an equilibrium. Here is a simple example. Man A has indifference curves for two goods that are like rectangular hyperbolas. So far so good. But Man B has indifference curves in terms of his consumption of those same goods which are like quarter circles, or at least are very slightly convex from above. This denies the usual textbook convexity, but what does B care about that! Now let us start each man out with a given endowment of both goods, and derive the resulting competitive supply and demand curves. Man A's will be of the normal continuous type. But Man B's demand curve will defy the continuity axiom of Wald or McKenzie. Figure 1 shows how the resulting demand curve may "have a hole in it" and make the existence of competitive equilibrium impossible. (The reader might imagine a servo that drives price up when demand exceeds supply, and argue that some kind of a statistical averaging

- 13 -

out occurs at what would be the equilibrium intersection of the continuous curve drawn from A to B to C to D.)

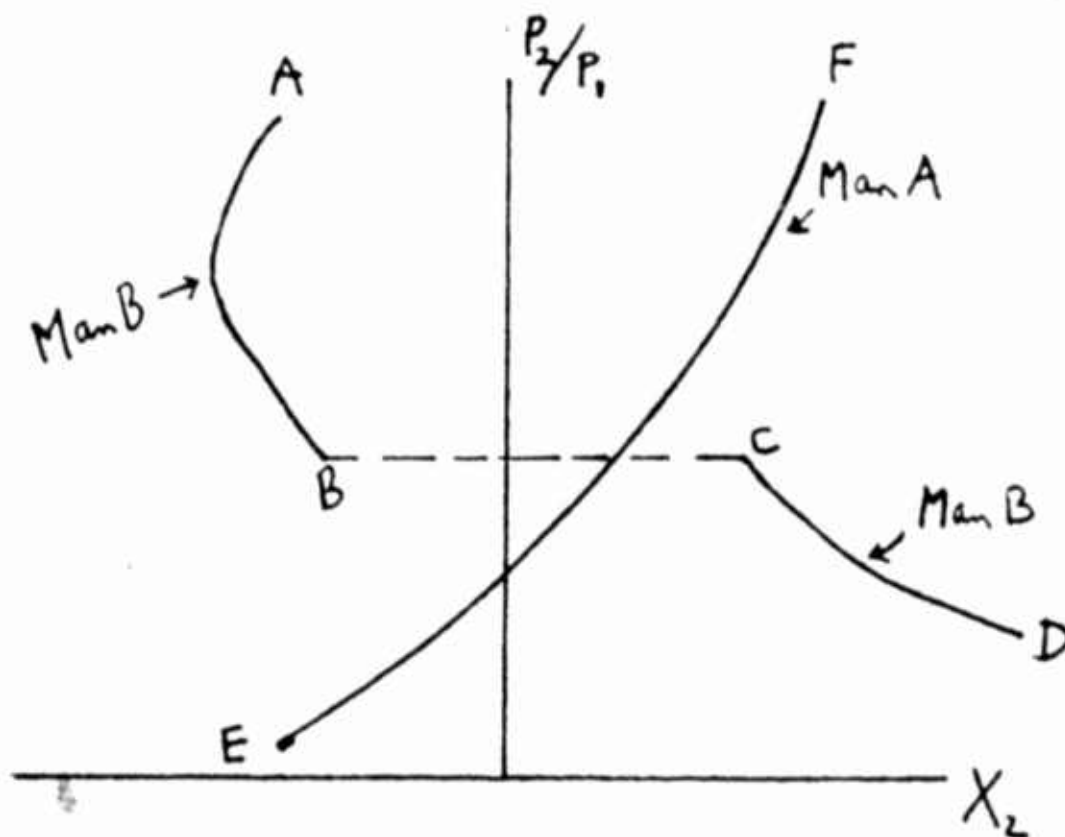


Figure 1



- 14 -

## V Power of Advanced Methods

Advanced mathematical methods are usually considered more difficult than elementary ones. The reverse is often the case. I shall conclude with an example which illustrates enormous simplification of the mathematical economist's technical task that results when he uses a few of the concepts of inequality and convexity rather than the intricate tools of the advanced calculus (such as Jacobians, Hessians, bordered determinants, definite quadratic forms, etc.)

First, consider the classical law of diminishing returns as applied to a smooth production function involving many variables. Such a function is usually assumed to be homogeneous of the first degree, so that

$$y = g(x_1, \dots, x_n) = g(x) = g(\lambda x) / \lambda \quad \text{for } \lambda > 0$$

$$(4) \quad g(x) = \sum_{j=1}^n x_j \frac{\partial g}{\partial x_j}$$

$$\sum_{j=1}^n \frac{\partial^2 g}{\partial x_i \partial x_j} x_j = 0, \quad \text{so that } H = \left[ \frac{\partial^2 g}{\partial x_i \partial x_j} \right] = H' \text{ is a singular matrix.}$$

To the assumption of constant-returns-to-scale is added the usual assumption of "diminishing returns to disproportionate changes." The mathematical economist summarizes this by requiring  $H$  to be negative semi-definite, usually of rank  $n - 1$ .

All this is expressed with more economical assumptions and exposition by the two requirements

$$(5)a \quad y = g(x) = g(\lambda x) / \lambda \quad , \lambda > 0$$

$$(5)b \quad g(x+z) \geq g(x) + y(z)$$

- 15 -

Indeed, we could dispense with the first of these conditions if we modify the second so as to require the equality sign when  $[x_1] = \alpha [x_1]$ .

From (5)a and (5)b we can easily deduce the fundamental convexity of the equal-product contours that is so important for classical competitive theory and for non-linear and linear programming: in words, a point half way between two points on the same contour can never lie below that contour. Or

$$g(a) = g(b) \text{ implies } g\left(\frac{1}{2}a + \frac{1}{2}b\right) \geq g(a) = g(b)$$

Proof:

$$g\left(\frac{1}{2}a + \frac{1}{2}b\right) = \frac{1}{2}g(a+b) \geq \frac{g(a) + g(b)}{2}$$

a relation which holds even if  $g(a) \neq g(b)$ .<sup>3</sup>

Without any modification, (5)b by itself serves to rule out decreasing returns to scale. (Mathematical proof:  $g(\lambda x) \geq 2g(x)$  for all  $x$  does imply  $g(\lambda x) \geq \lambda g(x)$  for all  $\lambda > 0$ .) The common sense of this is important for economics. If we can always at worst get the sum of independent production activities, increasing scale can never lower unit returns. This means that anti-trust policy cannot rely on diseconomies of large scale production to police competition if the different parts of General Motors are always capable of desisting from contaminating each other.

Both (5)a and (5)b are hypotheses that can be falsified by reality; they are hence not provable by logic. Yet (5)b will seem to many economists to have greater empirical plausibility than the more special assumption of constant returns to scale. It is, therefore, worth pointing out the disastrous analytic consequences both for linear programming and for the usual versions of non-linear programming if (5)b is affirmed and (5)a denied. We cannot then be sure that the equal-product contours have the classically

- 15A -

The reader might tediously prove this from (4) by bordering the Hessian  $H$  by  $\partial g / \partial x_1$ , or by solving  $y = g(x_1, \dots, x_n)$  for  $x_1 = G(x_2, \dots, x_n, y)$  and proving the positive-definiteness of the  $(n-1)^{\text{st}}$  Hessian  $\partial^2 g / \partial x_i \partial x_j =$

$$\partial B_{11} / \partial x_j = B_{j1} \partial B_{11} / \partial x_1 \quad \text{where } B_{11} = - (\partial g / \partial x_1) / (\partial g / \partial x_1) \text{ for } i > 1.$$

- 16 -

postulated convexity needed for competitive equilibrium and needed to insure a local maximum is indeed a maximum in the large.

A single example will make this clear:  $g = x_1^2 + x_2^2$  satisfies (5)b but does not have the convexity property  $g(a) = g(b) \geq g(\frac{1}{2}a + \frac{1}{2}b)$ , as can be seen from the quarter-circle product contours. (Note that  $g = \sqrt{x_1^2 + x_2^2}$  has the same contours and does satisfy (5)a; hence, it could not satisfy (5)b--and does not. If  $g = \sqrt{x_1^2 + x_2^2}$  is technologically feasible and our final production function  $g^*(x_1, x_2)$  is required to satisfy (5)b, then  $g^*(x_1, x_2) \geq \sqrt{x_1^2} + \sqrt{x_2^2} = x_1 + x_2$ , and will in fact  $= x_1 + x_2$ , which will be 'economically relevant production function.')

Here is an important final theorem with many applications. To prove it by manipulation of bordered Hessians would be tedious indeed.

Theorem: Let  $U = f(x_1, \dots, x_n) = f(x)$  have the property:

$f(a) = f(b)$  implies  $f(\frac{1}{2}a + \frac{1}{2}b) \geq f(a)$ ; and let  $m$  functions

$g_1(x_1, \dots, x_n) = g_1(x)$  each have the properties

$$g_1(\lambda x) = \lambda g_1(x)$$

$$g_1(x+s) \geq g_1(x)$$

and let  $U = f(x)$  to be a maximum subject to

$$g_i(x) \geq Y_i \quad (i=1, \dots, m)$$

attain a maximum equal to  $U = F(Y_1, \dots, Y_m) = F(Y)$ .

Then,  $F$  has the same property as  $f$ , namely

$$F(A) = F(B) \text{ implies } F(\frac{1}{2}A + \frac{1}{2}B) \geq F(A) = F(B)$$

Proof: Suppose  $f(a) = F(A) = f(b) = F(B)$  with  $g_1(a) \geq A$  and  $g_1(b) \geq B$ .

Then  $\frac{1}{2}a + \frac{1}{2}b$  is a feasible point in that  $g_1(\frac{1}{2}a + \frac{1}{2}b) = \frac{1}{2}g_1(a+b) \geq$

$$\frac{g_1(a)}{2} + \frac{g_1(b)}{2} = \frac{A+B}{2}. \quad \text{Certainly}$$

- 17 -

←  $f(\frac{1}{2}a + \frac{1}{2}b) \geq f(a) = f(b) = F(A) = F(B)$  by hypothesis, and since the optimal value  $F(\frac{1}{2}A + \frac{1}{2}B) \geq f(\frac{1}{2}a + \frac{1}{2}b)$ , our theorem is proved.

Special cases of this theorem are the following: (i) The efficient production possibility frontier relating total outputs and inputs must be a convex set if each production function satisfies the classical returns law. (Set  $G_1 = F_1(V_{11}, \dots, V_{n1}) - \sum_{j=1}^n V_{1j}$  to prove this.); (ii) In my equations (3),  $U = F(V_1, \dots, V_n) = F(V)$  has the stipulated convexity property; (iii) The beautiful Hicks theorem that composite goods have regularly shaped indifference contours follows if we set  $G_1 = P_1x_1 + \dots + P_r x_r$ ,  $G_2 = P_{r+1}x_{r+1} + \dots + P_k x_k$ , ... etc.

Lest the power of these methods betray us into ignoring the non-convexity that may occur in the real world, let me end with a valid inequality of revealed preference that holds even when we deny the convexity of the indifference curves needed for the above theorems. Whatever the shape of  $U = f(x_1, \dots, x_n)$ , truly maximising  $\sum_{j=1}^n P_j x_j$  subject to  $f(x) \geq \bar{U}$  will give us  $x_i = \psi_i(P_1, \dots, P_n)$  with the property  $\sum \Delta x_j \Delta P_j \leq 0$ —a result in revealed preference that is true even when many local maxima have to be painfully eliminated from the optimal solution. Other important examples in the realm of decreasing cost industries arise to plague the linear programmer and impatient economist.

Paul A. Samuelson

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